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Engineering Notes

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Near-Optimal Low-Thrust Lunar Trajectories

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Introduction

RECENTLY, several authors presented optimal trajectories for low-thrust spacecraft with the dynamics governed by the classical restricted three-body problem.[1–4] A common three-body low-thrust trajectory problem involves computing the minimum-fuel, planar transfer from circular, low Earth orbit (LEO) to circular, low lunar orbit (LLO). Although Refs. 1–4 involve a variety of solution methods (variational method, collocation, and hierarchal direct/indirect method), each method results in a numerically intense process because of the nature of low-thrust trajectories. That is, numerical integration of long-duration trajectories with numerous near-circular orbits about the Earth and moon is required. The boundary conditions and three-body dynamics further increase the sensitivity of the problem to the initial guesses for thrust control and initial orbit geometry.

In this Note, we present approximate solutions to a minimum-fuel problem for a planar LEO–LLO, low-thrust transfer that consists of a continuous-thrust Earth-escape spiral, a translunar coast arc, and a continuous-thrust moon-capture spiral. Our approximate solution method replaces the long-duration, many-revolution escape and capture spiral trajectories with curve fits of universal low-thrust trajectory solutions. This approach results in a very robust, quick, and efficient scheme for computing near-optimal, minimum-fuel LEO–LLO transfers for a wide range of thrust-to-weight (T/W) ratios. Numerical results from the approximate solution method are presented and the results compare very well with the corresponding true minimum-fuel LEO–LLO transfers.

Trajectory Optimization

Problem Statement

A complete problem statement for a minimum-fuel LEO–LLO transfer would involve finding the thrust direction time histories and the transfer times for the two powered spirals, the duration of the coast arc, and the initial Earth–moon geometry. The initial conditions are circular LEO and the terminal state conditions are circular LLO. The dynamics of the entire trajectory are governed by the classical restricted three-body problem.[5] A detailed problem statement for the true minimum-fuel LEO–LLO problem is presented in Ref. 3.

For our near-optimal, approximate solution method, the problem statement is presented as follows.

For the free end-time problem, find the powered Earth-escape spiral time $t_{\rm esc}$, the translunar coast time $t_{\rm coast}$, the powered moon-capture spiral time $t_{\rm capt}$, and the initial polar angle $\theta(0)$ that minimize

$$J = t_{\rm esc} + t_{\rm capt} \tag{1}$$

subject to the unpowered three-body equations of motion

$$\dot r = v_r \tag{2}$$

$$\dot v_r = \frac{v_\theta^2}{r} - \frac{\mu_e}{r^2} - \frac{\mu_m(r + D\cos\theta)}{(r^2 + 2Dr\cos\theta + D^2)^{\frac{3}{2}}} + \frac{\mu_m\cos\theta}{D^2} + 2\omega v_\theta + \omega^2 r \tag{3}$$

$$\dot v_\theta = \frac{\mu_m D\sin\theta}{(r^2 + 2Dr\cos\theta + D^2)^{\frac{3}{2}}} - \frac{\mu_m\sin\theta}{D^2} - 2\omega v_r - \frac{v_r v_\theta}{r} \tag{4}$$

$$\dot\theta = v_\theta / r \tag{5}$$

with the initial conditions at the start of the coast arc

$$r(0) = g_1(r_{\rm LEO}, \mu_e, a_T; t_{\rm esc}) \tag{6}$$

$$v_r(0) = g_2(r_{\rm LEO}, \mu_e, a_T; t_{\rm esc}) \tag{7}$$

$$v_\theta(0) = g_3(r_{\rm LEO}, \mu_e, a_T; t_{\rm esc}) \tag{8}$$

and the terminal state constraints at the end of the coast arc

$$r(t_{\rm coast}) = h_1(r_{\rm LLO}, \mu_m, a_T; t_{\rm capt}) \tag{9}$$

$$v_r(t_{\rm coast}) = h_2(r_{\rm LLO}, \mu_m, a_T; t_{\rm capt}) \tag{10}$$

$$v_\theta(t_{\rm coast}) = h_3(r_{\rm LLO}, \mu_m, a_T; t_{\rm capt}) \tag{11}$$

$$\theta(t_{\rm coast}) = \text{free} \tag{12}$$

Equations (2–5) are the three-body equations of motion for a coasting spacecraft in a rotating, Earth-centered polar coordinate frame. The x axis is along the Earth–moon line and is considered positive from the Earth center away from the moon. The states of the system are radial position r, radial velocity v_r, circumferential velocity v_θ, and polar angle θ. Polar angle θ is measured positive counterclockwise from the x axis. The gravitational parameters for the Earth and moon are denoted by μ_e and μ_m, respectively. The constant Earth–moon separation distance is D, and the constant angular rotation rate of the Earth–moon system is ω.

The initial and terminal states for the translunar coasting trajectory are provided by the functional relationships g_1–g_3 and h_1–h_3, which represent numerical curve fits of universal low-thrust spiral trajectory solutions. Perkins[6] developed approximate solutions for the motion of a low-thrust spacecraft in an inverse-square gravity field with a constant thrust acceleration vector pointed along the instantaneous velocity vector. These approximate solutions are independent of initial circular radius, thrust acceleration, or attracting body. Three nondimensional parametric curves with radial distance, velocity magnitude, and flight-path angle vs transfer time are constructed upon numerical integration of Perkins' approximate two-body equations of motion. These universal low-thrust solutions can be scaled by specifying the initial circular orbit radius, the gravitational parameter of the attracting body, and the constant thrust acceleration a_T of the spacecraft. Therefore, the vehicle's state at the end of the powered Earth-escape spiral trajectory (or start of the moon-capture spiral) is obtained by curve fitting the universal low-thrust solutions with transfer time $t_{\rm esc}$ (or $t_{\rm capt}$) as the independent variable. Next, the nondimensional states are scaled by using the appropriate initial circular radius, gravitational parameter, and thrust acceleration. This curve-fitting procedure is represented by Eqs. (6–11) and eliminates the need to numerically integrate the long-duration Earth-escape and moon-capture spirals. Furthermore, the approximation allows the simple computation of escape and capture spirals for a wide range of thrust levels. Perkins has demonstrated that the

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Table 1 Comparison between optimal and near-optimal lunar trajectory solutons

Case	$(T/W)_0$	r_{LEO} alt., km	r_{LLO} alt., km	t_{esc}, days	t_{coast}, days	t_{capt}, days	t_f, days	ΔV, km/s
Ref. 3 (posigrade)	3×10^{-3}	315	100	2.23	4.58	0.45	7.26	7.05
Near optimal	3×10^{-3}	315	100	2.25	4.57	0.44	7.27	7.09
Ref. 3 (retrograde)	3×10^{-3}	315	100	N/A	N/A	N/A	7.73	7.12
Near optimal	3×10^{-3}	315	100	2.26	5.03	0.46	7.75	7.15
Ref. 1 (posigrade)	1×10^{-3}	200	100	—	—	—	10.76	6.99
Near optimal	1×10^{-3}	200	100	6.37	4.70	1.14	12.20	7.43
Ref. 2 (retrograde)	2.3×10^{-4}	35,863	6710	N/A	N/A	N/A	15.90	2.23
Near optimal	2.3×10^{-4}	35,863	6710	9.36	4.31	2.53	16.20	2.35
Ref. 4 (posigrade)	1.3×10^{-4}	407	100	55.19	6.76	10.49	72.44	7.65
Near optimal	1.3×10^{-4}	407	100	55.48	7.52	10.55	73.56	7.66
Ref. 4 (posigrade)[a]	4.4×10^{-5}	407	100	161.55	7.06	31.39	200.00	7.69
Near optimal	4.4×10^{-5}	407	100	163.68	8.22	33.92	205.82	7.88

[a]Maximum payload solution with fixed $t_f = 200$ days.

universal low-thrust trajectory solutions nearly match exact numerically integrated trajectories for T/W ratios below 10^{-2} and that as T/W ratio decreases, the universal solutions become more accurate.

The objective of our optimization problem is to find the coasting trajectory between the two powered escape and capture spirals such that the sum of the total powered time ($t_{esc} + t_{capt}$) is minimized. Since the propellant mass flow rate is assumed to be constant, minimum total powered time corresponds to the minimum-fuel trajectory. The solution is termed near optimal since the two powered spiral trajectories are replaced by approximate two-body trajectories with tangential-thrust steering and only the coasting trajectory is governed by the three-body model.

Solution Method

The approximate LEO–LLO minimum-fuel problem is solved by using sequential quadratic programming (SQP), a direct parameter optimization method.[7] The SQP code used here computes the gradients with first-order forward differences and is due to Pouliot.[8] The optimization problem involves only four SQP design variables [t_{esc}, t_{coast}, t_{capt}, and $\theta(0)$] and three equality constraints (9–11). Numerical integration of the coasting trajectory is performed by a standard fourth-order, fixed-step, Runge–Kutta routine, and 500 integration steps were deemed sufficient for acceptable accuracy.

Although the problem consists of only four design variables, some initial simulations must be computed to determine a good initial guess for the SQP optimization code. The basic solution procedure is outlined as follows.

1) For a given initial T/W ratio and r_{LEO}, compute the apogee radius r_a for a range of Earth-escape spiral times t_{esc}. Apogee radius is calculated by using the engine cutoff state and two-body mechanics. Escape time t_{esc} is adjusted until r_a reaches between 45 and 50 Earth radii, which is the approximate distance to the lunar sphere of influence (SOI).

2) Next, the coasting trajectory is numerically integrated for a range of initial polar angle $\theta(0)$. Varying $\theta(0)$ essentially rotates the approximate Earth-escape spiral and the engine cutoff state. The coast time t_{coast} is set to a sufficiently large value (7–8 days) so that the time histories of the moon-relative states can be observed as the spacecraft approaches the moon and performs a flyby. Polar angle $\theta(0)$ is adjusted until a point in the translunar coasting trajectory enters the SOI with a negative moon-relative radial velocity component. Coast time can be estimated from the point where the SOI is initially entered.

3) Finally, the moon-capture time t_{capt} is adjusted until the moon-relative energy of the state from the approximate capture spiral nearly matches the energy from the end of the translunar coasting arc.

Numerical Results

Several near-optimal lunar trajectories are obtained for a wide range of initial T/W by using the approximate trajectory optimization method, and the results are presented in Table 1. Six separate optimal minimum-fuel trajectories from Refs. 1–4 are shown in Table 1, and the corresponding near-optimal transfers are tabulated below each respective optimal solution. The initial T/W ranges nearly two orders of magnitude from 3×10^{-3} to 4.4×10^{-5}. The performance measure of each trajectory is represented by the velocity increment ΔV, which is calculated by utilizing the final mass ratio and the rocket equation

$$\Delta V = -c\ell n(m_{LLO}/m_{LEO}) \tag{13}$$

where c is the engine exhaust velocity.

The near-optimal lunar trajectories display a very good match with the corresponding optimal trajectories from Refs. 1–4. As indicated by Table 1, both posigrade and retrograde LLO solutions are obtained. The first two near-optimal solutions for $T/W = 3 \times 10^{-3}$ exhibit nearly the same performance and transfer times as the minimum-fuel solutions from Ref. 3. Equivalent ΔV is only about 0.5% higher than the optimal performance for these two cases. Because the spacecraft from Ref. 1 is power limited, the thrust acceleration is variable and the optimal trajectory displays a drop in thrust acceleration during the translunar phase that essentially approximates a coasting arc. The near-optimal solution for a thrust-limited spacecraft cannot be directly compared to the results from Ref. 1, however; ΔV from the near-optimal solution is only 6.3% greater than the optimal power-limited trajectory. The retrograde lunar transfer from Ref. 2 is not actually from LEO to LLO since the initial circular Earth orbit is 35,863-km altitude (geosynchronous) and the final circular lunar orbit is 6,710-km altitude. The spacecraft from Ref. 2 has a constant thrust magnitude and a fixed thrust-coast-thrust engine sequence. Again, the near-optimal solution shows a good match with the minimum-fuel trajectory. Some numerical error between the approximate method and Refs. 1 and 2 can be attributed to slight differences in the constants used to model the three-body Earth–moon system. Finally, the near-optimal solutions for the two cases from Ref. 4 both exhibit a good match with the corresponding optimal trajectories. The second solution from Ref. 4 ($T/W = 4.4 \times 10^{-5}$) represents a maximum payload optimization problem with vehicle parameters (constant thrust magnitude and mass flow rate) as design variables and the end time fixed at 200 days. This maximum payload solution is equivalent to the minimum-fuel solution with fixed vehicle parameters and a fixed transfer time of 200 days. Therefore, to compute a comparative minimum-fuel trajectory with our approximate method, the vehicle parameters are fixed at the optimal values from the maximum payload solution. However, the minimum-transfer time solution from the approximate method results in $t_f = 203.86$ days and $\Delta V = 7.93$ km/s. Because the approximate method cannot complete the LEO–LLO transfer in fewer than 203 days, the free end-time, minimum-fuel, near-optimal solution in presented in Table 1, and the resulting transfer time is 205.82 days and $\Delta V = 7.88$ km/s.

As gravitational acceleration diminishes with distance from the attracting body, Perkins' approximate solutions approach straight-line asymptotes (near-vertical flight) for even very small initial a_T. This phenomenon has almost no effect on our lunar trajectories since the Earth and moon are relatively close and the spacecraft velocity for a typical low-thrust lunar transfer is well below parabolic speed.

Conclusions

An approximate method for computing near-optimal, minimum-fuel, planar lunar trajectories for low-thrust spacecraft has been developed. Our method approximates the long-duration powered Earth-escape and moon-capture spirals with curve fits from universal low-thrust trajectory solutions and numerically computes the translunar coasting trajectory between the curve-fit boundaries using the restricted three-body problem dynamics. The approximate method requires only four optimization variables, and solutions are readily obtained by using sequential quadratic programming. Near-optimal solutions for a wide range of initial thrust-to-weight ratios and initial and terminal circular orbit altitudes are obtained both quickly and with little computational load. The performance and orbit transfer characteristics of the near-optimal solutions exhibit a very close match with published optimal lunar trajectory solutions. This method would be a useful preliminary design tool for spacecraft and mission designers.

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Analytical Missile Guidance Laws with a Time-Varying Transformation

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I. Introduction

THE most popular homing missile guidance is based on a control law called proportional navigation.[1] The basic notion is that, if the line-of-sight rate is annulled, then (for a nonmaneuvering, constant-velocity target) the missile is on a collision course. If the target is considered smart or maneuvering, variations in the proportional navigation have been shown to result in better miss distances. These variations have been given optimal control foundations through linear quadratic Gaussian formulations.[2–5]

There are, however, a few problems with the use of such guidance laws. First, the measurements in an end game are nonlinear (bearing angle, range, and range rate) in Cartesian coordinates. As a consequence, there is linearization in the filtering update process. The measurements are linear in a polar coordinate–based state space. However, the propagation between the measurement updates in this case leads to nonlinear equations. Therefore, the states used in the guidance law are suboptimal. The second problem lies with the guidance law, which is formulated assuming separability of the guidance (control) law, and the estimators, which do not hold. It is usually formulated in Cartesian coordinates for linearity.[2,4] As a result, there is considerable scope for research in improving the missile performance in terms of estimator, guidance, and autopilot in an intercept scenario.[5]

This study is focused on obtaining improvements with a properly posed controller for guidance. Such a view will enable us to integrate the estimator in the loop in an optimal way. The central idea here is that the polar coordinates present a natural coordinate system for a missile engagement. A few papers in the literature[6,7] deal with guidance laws in polar coordinates, but their formulations do not derive an optimal guidance law as they do in this study. Lin and Tsai[8] and Rao,[9] in an extension of Ref. 8, consider an optimal control formulation seeking to maximize the final missile speed. In this study, we seek to minimize the control energy, and we consider a time-varying weight on the control to shape the relative kinematics. To obtain a closed-form solution for the commanded accelerations, the radial and transverse coordinates are decoupled. The decoupling of the coordinates leads to a two-point boundary-value problem with linear, time-varying coefficients. However, with a time-varying transformation, a class of closed-form solutions are obtained that yield several proportional guidance laws.

The rest of this Note is organized as follows: The optimal guidance problem is developed in polar coordinates in Sec. II. This problem is further shown to decompose into two decoupled optimal control problems, where a closed-form control solution is available in the radial direction and a time-varying linear dynamic system has to be solved for control in the transverse direction. A commonly used approximation for time-to-go and a transformation are shown to lead to a class of proportional navigation–type feedback guidance laws in Sec. III. The conclusions are summarized in Sec. IV.

II. Optimal Guidance in Decoupled Polar Coordinates

The dynamics of a target-intercept geometry are expressed by a set of coupled nonlinear differential equations in an inertial polar coordinate system as (Fig. 1)

$$\ddot{r} - r\dot{\theta}^2 = a_{T_r} - a_{M_r} \tag{1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_{T_\theta} - a_{M_\theta} \tag{2}$$

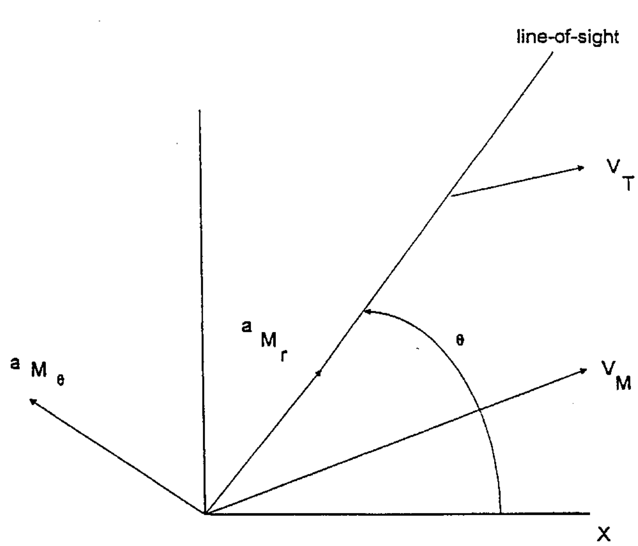

Fig. 1 Engagement geometry.

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